

MINIMAL FREE RESOLUTIONS OF 0-DIMENSIONAL SCHEMES IN $\mathbb{P}^1 \times \mathbb{P}^1$

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ABSTRACT. Let X be a zero-dimensional scheme in $\mathbb{P}^1 \times \mathbb{P}^1$. Then X has a minimal free resolution of length 2 if and only if X is ACM. In this paper we determine a class of reduced schemes whose resolutions, similarly to the ACM case, can be obtained by their Hilbert functions and depends only on their distributions of points in a grid of lines. Moreover, a minimal set of generators of the ideal of these schemes is given by curves split into the union of lines.

1. INTRODUCTION

Let $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a zero-dimensional scheme. Then it is well known that X has a minimal free resolution of length 2 if and only if X is ACM (see Giuffrida, Maggioni and Ragusa [2], Guardo and Van Tuyl [6] and Van Tuyl [11]). Other results about minimal free resolutions has been obtained for points in generic position (see Giuffrida, Maggioni and Ragusa [3] and [4]) and double points with ACM support in Guardo and Van Tuyl [6].

If X is an ACM zero-dimensional scheme, then its free resolution can be computed by looking at its Hilbert function and, at least if X is reduced, it depends only on the distribution of the points of X on a grid of $(1, 0)$ and $(0, 1)$ -lines. In this paper we study a class of reduced zero-dimensional schemes having these properties in common with ACM schemes. So in Theorem 3.6 we determine the Hilbert function and the minimal free resolution of these schemes and we show the connection between generators' and syzygies' degrees and the negative entries of the first difference of the Hilbert function. We also show that, similarly to ACM case, as minimal generators of the ideal of these schemes we can take curves split into the union of $(1, 0)$ and $(0, 1)$ -lines.

The starting point is Lemma 3.2 in which, given a zero-dimensional scheme Y and a point $P \in Y$, we determine the minimal free resolution of $Y \setminus \{P\}$, starting from the minimal free resolution of Y . This is proved under the conditions that P has just one minimal separator for Y and that the separator has a suitable degree. This result leads us to determine some reduced schemes whose minimal free resolution depends only on the distribution of these points on any grid of $(1, 0)$ and $(0, 1)$ -lines, similarly to what happens for ACM schemes. These schemes are obtained by erasing non collinear points of reduced ACM schemes and are the schemes studied in Theorem 3.6. In Example 3.7 we show that by deleting two collinear points of a reduced ACM schemes we get a scheme whose resolution is not as in Theorem 3.6 and there is no correspondence between generators and negative entries of the first difference of the Hilbert function.

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2. NOTATION AND PRELIMINARY RESULTS

Given an algebraically closed field k and given $\mathbb{P}^1 = \mathbb{P}_k^1$, let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ and let \mathcal{O}_Q be its structure sheaf. Let us consider the bi-graded ring $S = H_*^0 \mathcal{O}_Q = \bigoplus_{a,b \geq 0} H^0 \mathcal{O}_Q(a, b)$. For any bi-graded S -module N let $N_{i,j}$ be the component of degree (i, j) . For any $(i_1, j_1), (i_2, j_2) \in \mathbb{N}^2$ we write $(i_1, j_1) \geq (i_2, j_2)$ if $i_1 \geq i_2$ and $j_1 \geq j_2$ and we say that (i_1, j_1) and (i_2, j_2) are *comparable*. Given a zero-dimensional scheme $X \subset Q$, let $I_X \subset S$ be the associated saturated ideal and let $S(X) = S/I_X$.

Definition 2.1. The function $M_X: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{N}$ defined by:

$$M_X(i, j) = \dim_k S(X)_{i,j} = \dim_k S_{i,j} - \dim_k (I_X)_{i,j}$$

is called the *Hilbert function* of X . The function M_X can be represented as an infinite matrix with integers entries $M_X = (M_X(i, j)) = (m_{ij})$.

Note that $M_X(i, j) = 0$ for either $i < 0$ or $j < 0$ and so we restrict ourselves to the range $i \geq 0$ and $j \geq 0$. Moreover, for $i \gg 0$ and $j \gg 0$ $M_X(i, j) = \deg X$.

Definition 2.2. Given the Hilbert function M_X of a zero-dimensional scheme $X \subset Q$, the *first difference of the Hilbert function* of X is the matrix $\Delta M_X = (c_{ij})$, where $c_{ij} = m_{ij} - m_{i-1,j} - m_{i,j-1} + m_{i-1,j-1}$.

We consider the matrices $\Delta^R M_X = (a_{ij})$ and $\Delta^C M_X = (b_{ij})$, with $a_{ij} = m_{ij} - m_{i,j-1}$ and $b_{ij} = m_{ij} - m_{i-1,j}$. Note that $c_{ij} = a_{ij} - a_{i-1,j} = b_{ij} - b_{i,j-1}$, $m_{ij} = \sum_{h \leq i, k \leq j} c_{hk}$.

Theorem 2.3 ([2, Theorem 2.11]). *Given a zero-dimensional scheme $X \subset Q$ and given its Hilbert function M_X , the first difference $\Delta M_X = (c_{ij})$ satisfies the following conditions:*

- (1) $c_{ij} \leq 1$ and $c_{ij} = 0$ for $i \gg 0$ or $j \gg 0$;
- (2) if $c_{ij} \leq 0$, then $c_{rs} \leq 0$ for any $(r, s) \geq (i, j)$;
- (3) for every (i, j) $0 \leq \sum_{t=0}^j c_{it} \leq \sum_{t=0}^j c_{i-1,t}$ and $0 \leq \sum_{t=0}^i c_{tj} \leq \sum_{t=0}^i c_{t,j-1}$.

Remark 2.4. If $X \subset Q$ is a zero-dimensional scheme, let us consider $a = \min\{i \in \mathbb{N} \mid (I_X)_{i,0} \neq 0\} - 1$ and $b = \min\{j \in \mathbb{N} \mid (I_X)_{0,j} \neq 0\} - 1$. Then by Theorem 2.3 ΔM_X is zero out of the rectangle with opposite vertices $(0, 0)$ and (a, b) . In this case we say that ΔM_X is of size (a, b) .

Let $M_X = (m_{ij})$ be the Hilbert function of a zero-dimensional scheme $X \subset Q$. Using the notation in Giuffrida, Maggioni and Ragusa [2], for every $j \geq 0$ and $i \geq 0$ we set respectively:

$$\begin{aligned} i(j) &= \min\{t \in \mathbb{N} \mid m_{tj} = m_{t+1,j}\} = \min\{t \in \mathbb{N} \mid b_{t+1,j} = 0\} \\ j(i) &= \min\{t \in \mathbb{N} \mid m_{it} = m_{i,t+1}\} = \min\{t \in \mathbb{N} \mid a_{i,t+1} = 0\}. \end{aligned}$$

In particular, we see that $i(0) = a$ and $j(0) = b$.

For any zero-dimensional scheme $X \subset Q$ we have $1 \leq \text{depth } S(X) \leq 2$. X is called *arithmetically Cohen-Macaulay* (ACM for short) if $\text{depth } S(X) = 2$, in which case $S(X)$ is a Cohen-Macaulay ring.

Let $X \subset Q$ be a reduced ACM zero-dimensional scheme, let R_0, \dots, R_a and C_0, \dots, C_b be, respectively, $(1, 0)$ and $(0, 1)$ -lines containing X and each at least one point of X . Let $P_{ij} = R_i \cap C_j$ for any i, j . After a suitable permutation of $(1, 0)$

and $(0, 1)$ -lines, a graphical representation of X , inspired to the Ferrer's diagram (see, for example, Marino [9] and Van Tuyl [11]), is the following:

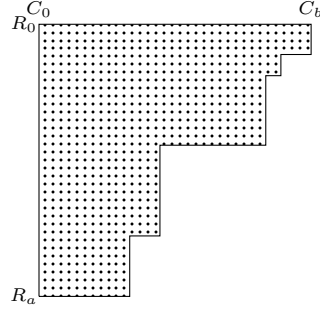


FIGURE 1. ACM scheme

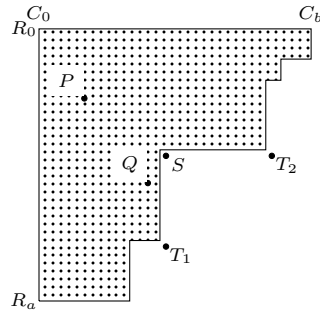
Theorem 2.5 ([2, Theorem 4.1]). *A zero-dimensional scheme $X \subset Q$ is ACM if and only if $c_{ij} \geq 0$ for any (i, j) .*

Let X be a reduced ACM zero-dimensional scheme X and $\Delta M_X = (c_{ij})$. By [1, Proposition 4.1] we see that $c_{ij} = 1$ if and only if $P_{ij} \in X$.

Definition 2.6. The pair (i, j) is called *corner* for X if $P_{i-1j}, P_{ij-1} \in X$, but $P_{ij} \notin X$. The pair (i, j) is called *vertex* for X if $P_{i-1j}, P_{ij-1} \notin X$ and $P_{i-1j-1} \in X$.

We give the following definitions:

Definition 2.7. A point $P_{ij} \in X$ is called *interior point* for X if there exists a corner $P_{rs} \in X$ such that $(i, j) < (r, s)$. A point $P_{ij} \in X$ is called *boundary point* for X if there is no corner $P_{rs} \in X$ such that $(i, j) < (r, s)$.



P = interior point, Q = boundary point, S = corner, T_1, T_2 = vertices

Remark 2.8. After a permutation of $(1, 0)$ and $(0, 1)$ -lines that preserves a configuration of points as in Figure 1, an interior point remains an interior point and a boundary point remains a boundary point. This can be shown as done in [9], noting that the definition of interior point is related to the definition of gap given by Marino in [9].

Now we recall the following definition:

Definition 2.9. Let $X \subset Q$ be a zero-dimensional scheme and let $P \in X$. The multiplicity of X in P , denoted by $m_X(P)$, is the length of $\mathcal{O}_{X,P}$.

Given $P \in Q$, we denote by I_P the maximal ideal of S associated to P . If $X \subset Q$ is a 0-dimensional scheme, then $I_X = \cap_{P' \in X} J_{P'}$ for some ideal $J_{P'}$ such that $\sqrt{J_{P'}} = I_{P'}$.

Definition 2.10. Given a zero-dimensional scheme $X \subset Q$ and $P \in X$ such that $m_X(P) = 1$, we say that $f \in S$ is a *separator* for $P \in X$ if $f(P) \neq 0$ and $f \in \cap_{P' \in X \setminus \{P\}} J_{P'}$. We say that f is a separator of *minimal separating degree* for $P \in X$ if f is a separator for $P \in X$ and if there are no separators g for $P \in X$ such that $\deg g < \deg f$. $\deg f$ is called *minimal separating degree* for P .

This definition generalizes the definition of a separator (see Orecchia [10]) for a point in a reduced zero-dimensional scheme in a multiprojective space given by Guardo, Marino and Van Tuyl [5] and Guardo and Van Tuyl [7].

Lemma 2.11. Let X be a zero-dimensional scheme and let $P \in X$ with $m_X(P) = 1$. Given $Z = X \setminus \{P\}$, (r, s) is the unique minimal separating degree for P if and only if:

$$H_Z(i, j) = \begin{cases} H_X(i, j) & \text{for } (i, j) \not\geq (r, s) \\ H_X(i, j) - 1 & \text{for } (i, j) \geq (r, s). \end{cases}$$

Proof. This follows by the exact sequence $0 \rightarrow \mathcal{I}_X \rightarrow \mathcal{I}_Z \rightarrow \mathcal{O}_P \rightarrow 0$. □

As a consequence of the previous lemma we see that, if P has just one minimal separating degree (r, s) for X , then a separator of minimal degree is unique modulo $(I_X)_{r,s}$ up to a scalar.

Remark 2.12. Let $X \subset Q$ be a reduced ACM zero-dimensional scheme and let $P \in X$. Suppose that $P = R \cap C$, with R and C , respectively, $(1, 0)$ and $(0, 1)$ -lines and let $p + 1 = \deg(X \cap R)$ and $q + 1 = \deg(X \cap C)$. Then by [8, Proposition 7.4] we see that (q, p) is the unique minimal separating degree for P in X .

Definition 2.13. Given a zero-dimensional scheme X , a pair (i, j) is called *corner* for $\Delta M_X = (c_{ij})$ if $c_{ij} \leq 0$ and $c_{ij_1} = c_{i-1j} = 1$. A pair (i, j) is called *vertex* for ΔM_X if $c_{i-1j} = c_{ij_1} \leq 0$ and $c_{i-1j-1} = 1$.

If X is an ACM zero-dimensional scheme, by the fact that $c_{ij} = 1$ if and only if $P_{ij} \in X$ it follows that (i, j) is a vertex (resp. corner) for ΔM_X if and only if P_{ij} is a vertex (resp. corner) for X . So:

- (1) if P is a boundary point, then $(q+1, p+1)$ is a vertex for ΔM_X and $c_{qp} = 1$;
- (2) if P is an interior point, then $(q+1, p+1)$ is not a vertex for ΔM_X and $c_{qp} = 0$.

So by Lemma 2.11 and by Theorem 2.5 it follows that the scheme $Z = X \setminus \{P\}$ is ACM if and only if P is a boundary point.

3. MINIMAL FREE RESOLUTIONS OF ZERO-DIMENSIONAL SCHEMES

Given a zero-dimensional scheme $X \subset Q$, we know that $1 \leq \text{depth } S(X) \leq 2$, so that $S(X)$ has a minimal free resolution of length ≤ 3 . $S(X)$ has a minimal free resolution of length 2 when X is ACM. In Giuffrida, Maggioni and Ragusa [2,

Example 3.1] there is a first example of a zero-dimensional scheme in Q that is not ACM. So we see that a minimal free resolution of a zero-dimensional scheme in Q is of the following type:

$$(1) \quad 0 \rightarrow \bigoplus_{i=1}^t \mathcal{O}_Q(-a_{3i}, -a'_{3i}) \rightarrow \bigoplus_{i=1}^n \mathcal{O}_Q(-a_{2i}, -a'_{2i}) \xrightarrow{\varphi} \bigoplus_{i=1}^m \mathcal{O}_Q(-a_{1i}, -a'_{1i}) \rightarrow \mathcal{I}_X \rightarrow 0$$

and X is ACM if $t = 0$. In particular, Giuffrida, Maggioni and Ragusa in [2, Theorem 4.1] show that, if X is ACM, then (a_{1i}, a'_{1i}) run over all the corners of X and (a_{2i}, a'_{2i}) run over all the vertices of X .

It is possible to have an exact sequence of length 2 that is not a resolution. Indeed, as shown in Giuffrida, Maggioni and Ragusa [2, Remark 3.2], we can have an exact sequence of type:

$$0 \rightarrow \mathcal{O}_Q(-r_1 - r_2, -s_1 - s_2) \rightarrow \mathcal{O}_Q(-r_1, -s_1) \oplus \mathcal{O}_Q(-r_2, -s_2) \rightarrow \mathcal{I}_X \rightarrow 0$$

where X is the intersection of two curves in Q , but it is not a complete intersection. In fact in Giuffrida, Maggioni and Ragusa [2, Theorem 1.2] we see that the only complete intersections in Q are obtained by intersecting two curves of type $(a, 0)$ and $(0, b)$. From a cohomological point of view the problem is that $h^1 \mathcal{O}_Q(i, j)$ can be nonzero. So the exact sequence (1) is a resolution if and only if the following conditions hold for any (r, s) :

- $H^0 \bigoplus_{i=1}^m \mathcal{O}_Q(r - a_{1i}, s - a'_{1i}) \rightarrow H^0 \mathcal{I}_X(r, s)$ is surjective
- $H^0 \bigoplus_{i=1}^n \mathcal{O}_Q(r - a_{2i}, s - a'_{2i}) \rightarrow H^0 \mathcal{F}(r, s)$ is surjective, where $\mathcal{F} = \text{Im } \varphi$.

In this way the sequence:

$$0 \rightarrow H^0 \bigoplus_{i=1}^t \mathcal{O}_Q(r - a_{3i}, s - a'_{3i}) \rightarrow H^0 \bigoplus_{i=1}^n \mathcal{O}_Q(r - a_{2i}, s - a'_{2i}) \xrightarrow{\varphi} H^0 \bigoplus_{i=1}^m \mathcal{O}_Q(r - a_{1i}, s - a'_{1i}) \rightarrow H^0 \mathcal{I}_X(r, s) \rightarrow 0$$

is exact for any (r, s) . In this section we compute the minimal free resolution of some particular zero-dimensional schemes in $\mathbb{P}^1 \times \mathbb{P}^1$.

Given a zero-dimensional scheme $X \subset Q$, let R_0, \dots, R_a and C_0, \dots, C_b be, respectively, the $(1, 0)$ and $(0, 1)$ -lines containing X and each at least one point of X . Let $P_{i,j} = R_i \cap C_j$ for any i, j .

Lemma 3.1. *Let X be an ACM zero-dimensional scheme and let $P_{hk} \in X$ be an interior point. Let $q + 1 = \#(X \cap C_k)$ and $p + 1 = \#(X \cap R_h)$. Given the minimal free resolution of X :*

$$0 \rightarrow \bigoplus_{i=1}^{m-1} \mathcal{O}_Q(-a_{2i}, -a'_{2i}) \xrightarrow{\varphi} \bigoplus_{i=1}^m \mathcal{O}_Q(-a_{1i}, -a'_{1i}) \rightarrow \mathcal{I}_X \rightarrow 0,$$

then $(q, p) \not\leq (a_{1i}, a'_{1i})$ for any $i = 1, \dots, m$ and $(q+1, p+1) \neq (a_{2i}, a'_{2i})$ for any $i = 1, \dots, m-1$.

Proof. If P_{hk} is an interior point, then there exist at least two vertices (r_1, s_1) and (r_2, s_2) for ΔM_X such that $(h, k) < (r_1, s_1)$ and $(h, k) < (r_2, s_2)$. This means that $(q, p) \geq (a_{2i}, a'_{2i})$ for some i , so that $(q+1, p+1) \neq (a_{2i}, a'_{2i})$ for any $i = 1, \dots, m-1$. \square

Lemma 3.2. *Let Y be a zero-dimensional scheme and let:*

$$0 \rightarrow \bigoplus_{i=1}^t \mathcal{O}_Q(-a_{3i}, -a'_{3i}) \rightarrow \bigoplus_{i=1}^n \mathcal{O}_Q(-a_{2i}, -a'_{2i}) \rightarrow \bigoplus_{i=1}^m \mathcal{O}_Q(-a_{1i}, -a'_{1i}) \rightarrow \mathcal{I}_Y \rightarrow 0$$

be the minimal free resolution of Y . Let $P \in Y$ with $m_Y(P) = 1$ and let $Z = Y \setminus \{P\}$. Suppose that there exist $r, s \in \mathbb{N}$ such that:

- (1) P has just one minimal separating degree (r, s) ;
- (2) $(r, s) \not\leq (a_{1i}, a'_{1i})$ for every $i = 1, \dots, m$;
- (3) $(r+1, s+1) \neq (a_{2i}, a'_{2i})$ for every $i = 1, \dots, n$.

Then the minimal free resolution of Z is:

$$\begin{aligned} (2) \quad 0 \rightarrow \bigoplus_{i=1}^t \mathcal{O}_Q(-a_{3i}, -a'_{3i}) \oplus \mathcal{O}_Q(-r-1, -s-1) \rightarrow \\ \rightarrow \bigoplus_{i=1}^n \mathcal{O}_Q(-a_{2i}, -a'_{2i}) \oplus \mathcal{O}_Q(-r-1, -s) \oplus \mathcal{O}_Q(-r, -s-1) \rightarrow \\ \rightarrow \bigoplus_{i=1}^m \mathcal{O}_Q(-a_{1i}, -a'_{1i}) \oplus \mathcal{O}_Q(-r, -s) \rightarrow \mathcal{I}_Z \rightarrow 0. \end{aligned}$$

Proof. By hypothesis P has just one minimal separating degree for Y , i.e. there exists just one curve containing Z and not Y , and this separator is a (r, s) -curve F . So we get the exact sequence:

$$0 \rightarrow \mathcal{I}_P(-r, -s) \rightarrow \mathcal{I}_Y \rightarrow \mathcal{I}_{Z|F} \rightarrow 0,$$

with $\mathcal{I}_{Z|F}$ ideal sheaf of Z in \mathcal{O}_F , and by hypothesis and by Lemma 2.11 we see that:

- (i) the map $H^0 \mathcal{I}_Y(i, j) \rightarrow H^0 \mathcal{I}_{Z|F}(i, j)$ is surjective for any (i, j) ;
- (ii) the minimal generators of Y are minimal generators of Z , because $(r, s) \not\leq (a_{1i}, a'_{1i})$ for every i .

Let us consider the mapping cone on this sequence:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & \bigoplus_{i=1}^t \mathcal{O}_Q(-a_{3i}, -a'_{3i}) & & \\
 & & & & \downarrow & & \\
 & & & & \bigoplus_{i=1}^n \mathcal{O}_Q(-a_{2i}, -a'_{2i}) & & \\
 & & & & \downarrow & & \\
 & & & & \bigoplus_{i=1}^m \mathcal{O}_Q(-a_{1i}, -a'_{1i}) & & \\
 & & & & \downarrow & & \\
 0 & \longrightarrow & \mathcal{I}_P(-r, -s) & \longrightarrow & \mathcal{I}_Y & \longrightarrow & \mathcal{I}_{Z|F} \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

and we get the exact sequence:

$$\begin{aligned}
 0 \rightarrow \bigoplus_{i=1}^t \mathcal{O}_Q(-a_{3i}, -a'_{3i}) \oplus \mathcal{O}_Q(-r-1, -s-1) \rightarrow \\
 \rightarrow \bigoplus_{i=1}^n \mathcal{O}_Q(-a_{2i}, -a'_{2i}) \oplus \mathcal{O}_Q(-r-1, -s) \oplus \mathcal{O}_Q(-r, -s-1) \rightarrow \\
 \rightarrow \bigoplus_{i=1}^m \mathcal{O}_Q(-a_{1i}, -a'_{1i}) \rightarrow \mathcal{I}_{Z|F} \rightarrow 0.
 \end{aligned}$$

Since $\deg F = (r, s)$, we get the exact sequence (2). By (i) and (ii) we see that the minimal generators of Z have degrees (a_{1i}, a'_{1i}) for $i = 1, \dots, m$ and (r, s) . By hypothesis and by Lemma 2.11 we see also that the degrees of the first syzygies of I_Z are among $(r+1, s)$, $(r, s+1)$ and (a_{2i}, a'_{2i}) for $i = 1, \dots, n$. By the fact that $(r+1, s+1) \neq (a_{2i}, a'_{2i})$ for every i , $(r+1, s+1)$ does not cancel out with any (a_{2i}, a'_{2i}) and by the construction of the mapping cone (a_{3i}, a'_{3i}) does not cancel out any of the (a_{2i}, a'_{2i}) . If (a_{3i}, a'_{3i}) cancels out either $(r+1, s)$ or $(r, s+1)$ for some i , then $(r+1, s+1) > (a_{3i}, a'_{3i})$ and some second syzygies regarding the generators of Y disappear. So we deduce that the sequence (2) is the minimal free resolution of \mathcal{I}_Z . \square

Corollary 3.3. *Let $X \subset Q$ be an ACM zero-dimensional scheme and let $P = R \cap C \in X$ be an interior point such that $m_X(P) = 1$, with R and C $(1, 0)$ and $(0, 1)$ -lines, respectively. Let:*

$$0 \rightarrow \bigoplus_{i=1}^{m-1} \mathcal{O}_Q(-a_{2i}, -a'_{2i}) \rightarrow \bigoplus_{i=1}^m \mathcal{O}_Q(-a_{1i}, -a'_{1i}) \rightarrow \mathcal{I}_X \rightarrow 0$$

be the minimal free resolution of X . Then the minimal free resolution of $Z = X \setminus \{P\}$ is:

$$\begin{aligned} 0 \rightarrow \mathcal{O}_Q(-q-1, -p-1) \rightarrow \\ \rightarrow \bigoplus_{i=1}^{m-1} \mathcal{O}_Q(-a_{2i}, -a'_{2i}) \oplus \mathcal{O}_Q(-q, -p-1) \oplus \mathcal{O}_Q(-q-1, -p) \rightarrow \\ \rightarrow \bigoplus_{i=1}^m \mathcal{O}_Q(-a_{1i}, -a'_{1i}) \oplus \mathcal{O}_Q(-q, -p) \rightarrow \mathcal{I}_Z \rightarrow 0, \end{aligned}$$

with $q+1 = \#(X \cap C)$ and $p+1 = \#(X \cap R)$.

Proof. By Marino [8, Proposition 7.4] we see that there exists just one minimal separating degree (q, p) and that a separator for P of X of minimal degree (q, p) splits into the union of linear forms. So we can proceed as in Lemma 3.2 and then the conclusion follows by Lemma 3.1. \square

Now we prove the following:

Theorem 3.4. *Let X be a reduced ACM zero-dimensional scheme and let $P_{i_1 j_1}, \dots, P_{i_h j_h} \in X$ be interior points such that $i_1 \neq \dots \neq i_h$ and $j_1 \neq \dots \neq j_h$. Let us consider:*

$$Z = X \setminus \{P_{i_1 j_1}, \dots, P_{i_h j_h}\}.$$

If the minimal free resolution of X is:

$$0 \rightarrow \bigoplus_{i=1}^{m-1} \mathcal{O}_Q(-a_{2i}, -a'_{2i}) \rightarrow \bigoplus_{i=1}^m \mathcal{O}_Q(-a_{1i}, -a'_{1i}) \rightarrow \mathcal{I}_X \rightarrow 0,$$

and $q_l + 1 = \#(X \cap C_l)$ and $p_l + 1 = \#(X \cap R_l)$ for $l = 1, \dots, h$, then the minimal free resolution of Z is:

$$\begin{aligned} 0 \rightarrow \bigoplus_{l=1}^h \mathcal{O}_Q(-q_l - 1, -p_l - 1) \rightarrow \\ \rightarrow \bigoplus_{i=1}^{m-1} \mathcal{O}_Q(-a_{2i}, -a'_{2i}) \oplus \bigoplus_{l=1}^h \mathcal{O}_Q(-q_l, -p_l - 1) \oplus \bigoplus_{l=1}^h \mathcal{O}_Q(-q_l - 1, -p_l) \rightarrow \\ \rightarrow \bigoplus_{i=1}^m \mathcal{O}_Q(-a_{1i}, -a'_{1i}) \oplus \bigoplus_{l=1}^h \mathcal{O}_Q(-q_l, -p_l) \rightarrow \mathcal{I}_Z \rightarrow 0. \end{aligned}$$

In order to prove Theorem 3.4 we need to remark that it is always possible to suppose that one of the following conditions holds:

- (1) $q_l < q_{l+1}$
- (2) $q_l = q_{l+1}$ and $p_l \leq p_{l+1}$

for any $l = 1, \dots, h$. We also need the following result:

Proposition 3.5. *Given $Z_l = X \setminus \{P_{i_1 j_1}, \dots, P_{i_l j_l}\}$, for $l = 1, \dots, h-1$, then:*

$$H_{Z_{l+1}}(i, j) = \begin{cases} H_{Z_l}(i, j) & \text{for } (i, j) \not\geq (q_{l+1}, p_{l+1}) \\ H_{Z_l}(i, j) - 1 & \text{for } (i, j) \geq (q_{l+1}, p_{l+1}). \end{cases}$$

Proof. It is sufficient to apply recursively [2, Lemma 2.15] and [1, Theorem 3.1]. \square

proof of Theorem 3.4. We prove the statement by induction on h . If $h = 1$, then it follows by Corollary 3.3.

Let $h > 1$ and suppose that $Z_{h-1} = X \setminus \{P_{i_1 j_1}, \dots, P_{i_{h-1} j_{h-1}}\}$ has the following resolution:

$$\begin{aligned} 0 \rightarrow \bigoplus_{l=1}^{h-1} \mathcal{O}_Q(-q_l - 1, -p_l - 1) \rightarrow \\ \rightarrow \bigoplus_{i=1}^{m-1} \mathcal{O}_Q(-a_{2i}, -a'_{2i}) \oplus \bigoplus_{l=1}^{h-1} \mathcal{O}_Q(-q_l, -p_l - 1) \oplus \bigoplus_{l=1}^{h-1} \mathcal{O}_Q(-q_l - 1, -p_l) \rightarrow \\ \rightarrow \bigoplus_{i=1}^m \mathcal{O}_Q(-a_{1i}, -a'_{1i}) \oplus \bigoplus_{l=1}^{h-1} \mathcal{O}_Q(-q_l, -p_l) \rightarrow \mathcal{I}_Z \rightarrow 0. \end{aligned}$$

To prove the statement we need to verify the conditions of Lemma 3.2. By Proposition 3.5 and Lemma 2.11 we see that $P_{i_h j_h}$ has just one minimal separating degree (q_h, p_h) for Z_{h-1} . Since $P_{i_h j_h}$ is an interior point, then (i_h, j_h) is comparable with two vertices and so (q_h, p_h) is greater or equal than a corner. This implies that $(q_h, p_h) \not\prec (a_{1i}, a'_{1i})$ for any $i = 1, \dots, m$ and $(q_h + 1, p_h + 1) \neq (a_{2i}, a'_{2i})$ for any $i = 1, \dots, m - 1$. By the fact that one of these conditions holds:

- (1) $q_h > q_{h-1}$
- (2) $q_{h-1} = q_h - 1$ and $p_h > p_{h-1}$

we see that $(q_h, p_h) \not\prec (q_l, p_l)$ and $(q_h + 1, p_h + 1) \neq (q_l, p_l + 1), (q_l + 1, p_l)$ for any $l = 1, \dots, h - 1$. So we can apply Lemma 3.2 and the theorem is proved. \square

If X is any ACM zero-dimensional scheme, then Giuffrida, Maggioni and Ragusa in [2, Theorem 4.1] show that the Hilbert function of X , precisely its first difference, determines the degrees of the generators and the first syzygies of X , so that ΔM_X determines the minimal free resolution of X . In fact, the degrees of the generators are equal to the corners for ΔM_X and those of the first syzygies are equal to the vertices of ΔM_X . Moreover, the minimal generators of X split into the union of $(1, 0)$ and $(0, 1)$ -lines. For the schemes Z given in Theorem 3.4 we have an analogous result.

Theorem 3.6. *Let X be a reduced ACM zero-dimensional scheme and let $P_{i_1 j_1}, \dots, P_{i_h j_h} \in X$ be interior points such that $i_1 \neq \dots \neq i_h$ and $j_1 \neq \dots \neq j_h$. Let us consider:*

$$Z = X \setminus \{P_{i_1 j_1}, \dots, P_{i_h j_h}\}.$$

Let $q_l + 1 = \#(X \cap C_l)$ and $p_l + 1 = \#(X \cap R_l)$ for $l = 1, \dots, h$ and let:

$$r_{ij} = \#\{l \in \{1, \dots, h\} \mid (q_l, p_l) = (i, j)\},$$

for any (i, j) . Then the minimal free resolution of Z is:

$$\begin{aligned}
0 \rightarrow \bigoplus_{l=1}^h \mathcal{O}_Q(-q_l - 1, -p_l - 1) \rightarrow \\
\rightarrow \bigoplus_{i=1}^{m-1} \mathcal{O}_Q(-a_{2i}, -a'_{2i}) \oplus \bigoplus_{l=1}^h \mathcal{O}_Q(-q_l, -p_l - 1) \oplus \bigoplus_{l=1}^h \mathcal{O}_Q(-q_l - 1, -p_l) \rightarrow \\
\rightarrow \bigoplus_{i=1}^m \mathcal{O}_Q(-a_{1i}, -a'_{1i}) \oplus \bigoplus_{l=1}^h \mathcal{O}_Q(-q_l, -p_l) \rightarrow \mathcal{I}_Z \rightarrow 0,
\end{aligned}$$

there exists a minimal set of generators of I_Z given by curves split into the union of $(1, 0)$ and $(0, 1)$ -lines and

$$\Delta M_Z(i, j) = \Delta M_X(i, j) - r_{ij}$$

for any (i, j) . In particular, we see that a pair (i, j) is the degree of:

- (1) a minimal generator for Z if and only if one of the following conditions holds:
 - (a) (i, j) is a corner for ΔM_Z
 - (b) $c_{ij} < 0$
- (2) a first syzygy for Z if and only if one of the following conditions holds:
 - (a) (i, j) is a vertex for ΔM_Z
 - (b) $c_{ij-1} < 0$
 - (c) $c_{i-1j} < 0$
- (3) a second syzygy if and only if $c_{i-1j-1} < 0$.

Proof. The statements follows easily by Proposition 3.5 and Theorem 3.4. We only need to remark that, in the proof of Proposition 3.5, there exists a minimal separator of $P_{i_{l+1}j_{l+1}}$ for Z_l that splits into the union of linear forms. So by the mapping cone procedure used we get a minimal set of generators of I_Z , each one of them split into the union of $(1, 0)$ and $(0, 1)$ -lines. \square

Example 3.7. In this example we show that Theorem 3.6 does not hold if we admit the possibility that there are two points $P_{i_{k_1}j_{k_1}}$ and $P_{i_{k_2}j_{k_2}}$ such that either $i_{k_1} = i_{k_2}$ or $j_{k_1} = j_{k_2}$. For example, consider the following reduced ACM zero-dimensional scheme X :

$$\begin{array}{cccc}
& C_0 & C_1 & C_2 & C_3 \\
R_0 & \bullet & \bullet & \bullet & \bullet \\
R_1 & \bullet & \bullet & &
\end{array}$$

and, given $P_{00} = R_0 \cap C_0$ and $P_{01} = R_0 \cap C_1$, let $Z = X \setminus \{P_{00}, P_{01}\}$. Then P_{01} has for X just one minimal separating degree that is $(1, 3)$ and a separator corresponds to the curve $R_1 \cup C_0 \cup C_2 \cup C_3$. The point P_{00} for $X \setminus \{P_{01}\}$ has also just one minimal separating degree that is $(1, 2)$ and a minimal separator corresponds to the curve $R_1 \cup C_2 \cup C_3$. So Z cannot have $R_1 \cup C_0 \cup C_2 \cup C_3$ and $R_1 \cup C_2 \cup C_3$ as minimal generators and the statement in Theorem 3.6 does not hold. Moreover by [1, Theorem 4.2] we get that the first difference of M_Z is the following:

and its minimal free resolution is:

$$\begin{aligned}
0 \rightarrow \mathcal{O}_Q(-6, -3) \oplus \mathcal{O}_Q(-6, -7) \oplus \mathcal{O}_Q(-5, -5) \oplus \mathcal{O}_Q(-4, -7)^{\oplus 2} \rightarrow \\
\rightarrow \mathcal{O}_Q(-6, -2) \oplus \mathcal{O}_Q(-5, -3) \oplus \mathcal{O}_Q(-4, -5) \oplus \mathcal{O}_Q(-3, -7) \oplus \\
\oplus \mathcal{O}_Q(-5, -3) \oplus \mathcal{O}_Q(-6, -2) \oplus \mathcal{O}_Q(-5, -7) \oplus \mathcal{O}_Q(-6, -6) \oplus \\
\oplus \mathcal{O}_Q(-5, -4) \oplus \mathcal{O}_Q(-4, -5) \oplus \mathcal{O}_Q(-4, -6)^{\oplus 2} \oplus \mathcal{O}_Q(-3, -7)^{\oplus 2} \rightarrow \\
\rightarrow \mathcal{O}_Q(-6, 0) \oplus \mathcal{O}_Q(-5, -2)^{\oplus 2} \oplus \mathcal{O}_Q(-4, -3) \oplus \mathcal{O}_Q(-3, -5) \oplus \mathcal{O}_Q(0, -7) \oplus \\
\oplus \mathcal{O}_Q(-5, -6) \oplus \mathcal{O}_Q(-4, -4) \oplus \mathcal{O}_Q(-3, -6)^{\oplus 2} \rightarrow \mathcal{I}_Z \rightarrow 0.
\end{aligned}$$

In particular, the minimal generators of X of degrees $(6, 0)$, $(5, 2)$, $(4, 3)$, $(3, 5)$ and $(0, 7)$ are minimal generators of Z too. The other minimal generators of Z are in degrees $(5, 2)$, $(4, 4)$, $(3, 6)$ (two in this degree) and $(5, 6)$ and they correspond to the minimal separating degree of the points $P_{04}, P_{13}, P_{21}, P_{32}, P_{40}$ for X . Moreover, each of these points have a separator that corresponds to a curve split in the union of lines.

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